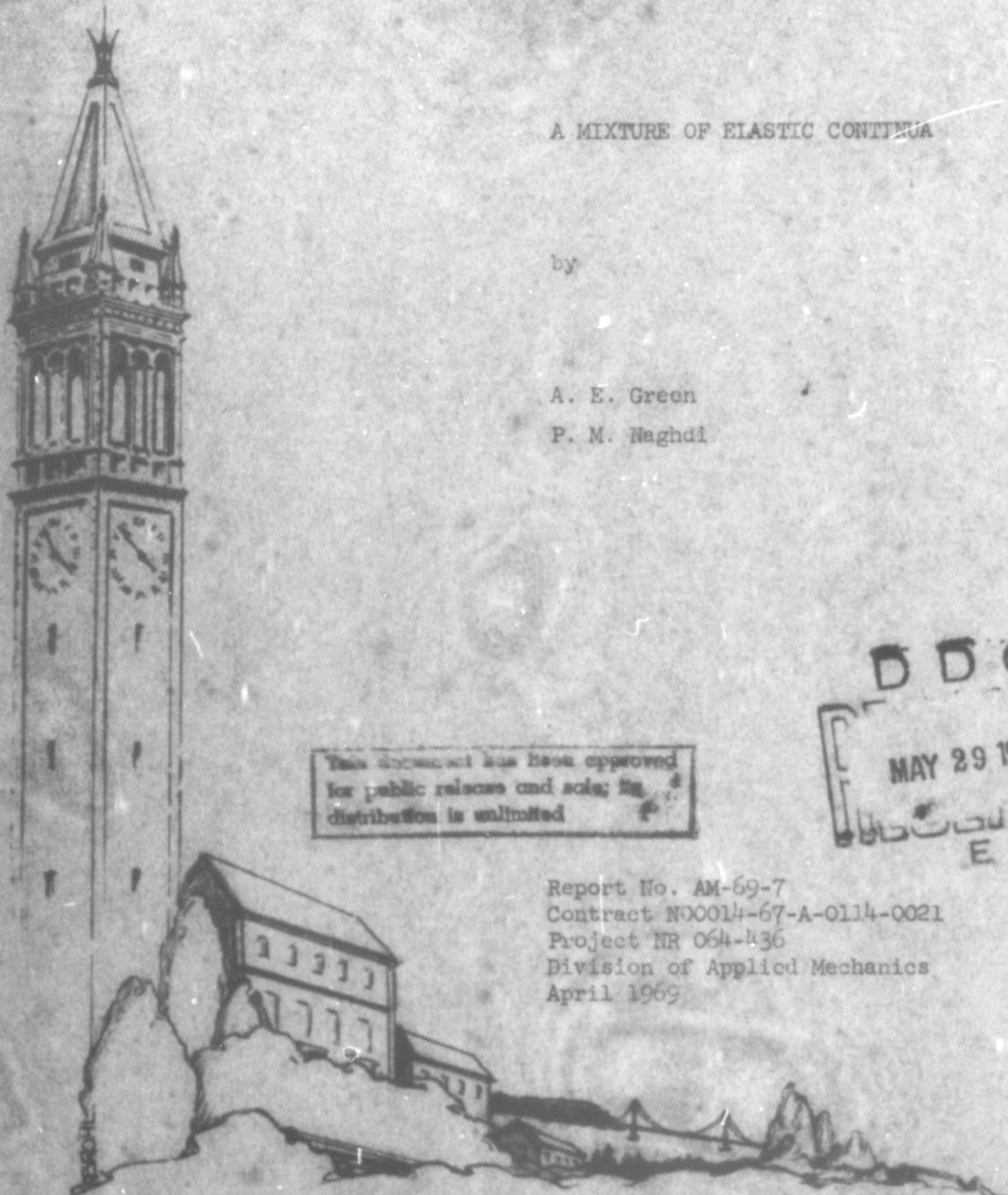

A MIXTURE OF ELASTIC CONTINUA

by

A. E. Green
P. M. Naghdi

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A. E. Green* and P. M. Naghdi+

1. Introduction

Mixtures of chemically inert elastic continua have been discussed previously by a number of writers. It is the purpose of the present note to provide a fairly simple discussion of the constitutive equations for a mixture of ν elastic continua with a single temperature.

Green and Naghdi [1] have recently made a small change in the use of their original theory of interacting continua [2] involving a single temperature. On the basis of this theory, we consider here a mixture of ν chemically inert elastic continua whose constitutive equations are nonlinear functions of temperature and suitable kinematical variables of each constituent but are linear functions of degree one in temperature gradient and velocity differences. The results obtained have a simple form.

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2. Notation and preliminaries

We consider a mixture of ν interacting constituents, each of which is regarded as a continuum. We assume that each point within the mixture is occupied simultaneously by all ν constituents which are in motion relative to a fixed system of rectangular Cartesian axes. The position of a typical particle of the α^{th} constituent at time τ is denoted by $x_i^{\alpha}(\tau)$, where

$$x_i^{\alpha}(\tau) = x_i^{\alpha}(X_1^{\alpha}, X_2^{\alpha}, X_3^{\alpha}, \tau) \qquad (-\infty < \tau \leq t) \quad , \tag{2.1}$$

and X_i^{α} is a reference position of the particles of this constituent. All Latin indices are tensor indices and take the values 1,2,3 and the usual summation convention is employed. The Greek letter α , as in (2.1), is reserved for reference to the α^{th} constituent of the mixture only. Unless indicated otherwise the summation convention does not apply to Greek indices. We use the notation

$$x_i^{\alpha} = x_i^{\alpha}(t) \quad , \tag{2.2}$$

and assume that the particles of each constituent all occupy the same position at time t . We refer to this position at time t as x_i and write

$$x_i^1 = x_i^2 = \quad = x_i \quad . \tag{2.3}$$

The velocity vectors at the point x_i at time t are given by

$$v_i^{\alpha} = \frac{\overset{\alpha}{D}\, x_i^{\alpha}}{Dt} \qquad (\alpha = 1,2,\ldots,\nu) \quad , \tag{2.4}$$

where $\overset{\alpha}{D}/Dt$ denotes differentiation with respect to t holding X_k^{α} fixed in the α^{th} continuum. This operator may be written in the form

$$\frac{\overset{\alpha}{D}}{Dt} = \frac{\partial}{\partial t} + v_m^{\alpha} \frac{\partial}{\partial x_m} \quad . \tag{2.5}$$

Acceleration vectors at time t are

$$f_i^{\alpha} = \frac{\overset{\alpha}{D}\, v_i^{\alpha}}{Dt} = \frac{\partial v_i^{\alpha}}{\partial t} + v_m^{\alpha} \frac{\partial v_i^{\alpha}}{\partial x_m} \qquad (\alpha = 1,2,\ldots,\nu) \quad , \tag{2.6}$$

the mass densities at time t are ρ_{α} and

$$m_{\alpha} = \frac{\partial \rho_{\alpha}}{\partial t} + \frac{\partial}{\partial x_k} (\rho_{\alpha}\, v_k^{\alpha}) \quad . \tag{2.7}$$

We define total density ρ and mean velocity v_i by the equations

$$\rho = \sum_{\alpha=1}^{\nu} \rho_{\alpha} \quad , \tag{2.8}$$

$$\rho v_i = \sum_{\alpha=1}^{\nu} \rho_{\alpha}\, v_i^{\alpha} \quad . \tag{2.9}$$

We also put

$$(\dot{\ }) = \frac{D(\ )}{Dt} = \frac{\partial(\ )}{\partial t} + v_m \frac{\partial(\ )}{\partial x_m} \quad , \tag{2.10}$$

and observe that

$$\frac{\overset{\alpha}{D}}{Dt} = \frac{D}{Dt} + \sum_{\alpha=1}^{\nu} \frac{\rho_\beta}{\rho} (v_k^\alpha - v_k^\beta) \frac{\partial}{\partial x_k} \quad . \tag{2.11}$$

Deformation gradients for each constituent are defined by

$$F_{ij}^\alpha = \frac{\partial x_i^\alpha}{\partial X_j^\alpha} \quad . \tag{2.12}$$

The residual energy equation for the mixture given by Mills [3] and recast in a slightly different form by Green and Naghdi [1] can be written as[+]

$$\rho r - q_{k,k} - \rho \frac{DU}{Dt} - \Phi + \sum_{\alpha=1}^{\nu} (\pi_i^\alpha v_i^\alpha + \sigma_{ki}^\alpha v_{i,k}^\alpha) = 0 \quad , \tag{2.13}$$

where a comma denotes partial differentiation with respect to x_k holding t fixed, r is the heat supply function, q_k the heat flux vector and σ_{ki}^α are partial stresses. Also

$$U = \sum_{\alpha=1}^{\nu} \frac{\rho_\alpha}{\rho} U_\alpha \quad , \quad \Phi = \sum_{\alpha=1}^{\nu} \frac{\partial}{\partial x_k} [\rho_\alpha u_k^\alpha U_\alpha] \quad , \quad u_k^\alpha = v_k^\alpha - v_k \quad , \tag{2.14}$$

where U_α is the internal energy per unit mass of the constituent α allowing for all interactions between this and the other constituents. In addition

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[+] Our equations of motion and energy and our entropy inequality can be shown to be equivalent to those given by Truesdell and Toupin [4] and Truesdell [5].

$$\frac{\partial \sigma^{\alpha}_{ki}}{\partial x_k} + \rho_\alpha(F^{\alpha}_i - f^{\alpha}_i) = \pi^{\alpha}_i + \frac{1}{2} m_\alpha(v^{\alpha}_i - v^{\nu}_i) \qquad (\alpha \neq \nu) \quad ,$$

$$\frac{\partial \sigma^{\nu}_{ki}}{\partial x_k} + \rho_\nu(F^{\nu}_i - f^{\nu}_i) = \pi^{\nu}_i + \frac{1}{2} \sum_{\alpha=1}^{\nu} m_\alpha(v^{\alpha}_i - v^{\nu}_i) \quad , \tag{2.15}$$

where F^{α}_i are body forces per unit mass and

$$\sum_{\alpha=1}^{\nu} \pi^{\alpha}_i = 0 \quad , \quad \sum_{\alpha=1}^{\nu} (\sigma^{\alpha}_{ki} - \sigma^{\alpha}_{ik}) = 0 \quad , \quad \sum_{\alpha=1}^{\nu} m_\alpha = 0 \quad . \tag{2.16}$$

We observe that

$$\rho \frac{DU}{Dt} + \Phi = \sum_{\alpha=1}^{\nu} (\rho_\alpha \frac{\overset{\alpha}{D} U_\alpha}{Dt} + m_\alpha U_\alpha) \quad . \tag{2.17}$$

If $T(> 0)$ is the temperature and S_α entropy per unit mass of the constituent α of the mixture, then the entropy inequality is given by

$$\rho T \frac{DS}{Dt} + T\,\Psi - \rho r + T(\frac{q_k}{T})_{,k} \geq 0 \quad , \tag{2.18}$$

where

$$\rho S = \sum_{\alpha=1}^{\nu} \rho_\alpha S_\alpha \quad , \quad \Psi = \sum_{\alpha=1}^{\nu} \frac{\partial}{\partial x_k} (\rho_\alpha u^{\alpha}_k S_\alpha) \quad . \tag{2.19}$$

Let A_α, A be the free energy functions defined by

$$A_\alpha = U_\alpha - TS_\alpha \quad , \quad \rho A = \sum_{\alpha=1}^{\nu} \rho_\alpha A_\alpha = \rho(U - TS) \quad . \tag{2.20}$$

Then, from (2.13) and (2.18), we have

$$-\rho\left(\frac{DA}{Dt} + S\,\frac{DT}{Dt}\right) - \Theta - \frac{\overset{*}{q}_k}{T}\frac{\partial T}{\partial x_k} + \sum_{\alpha=1}^{\nu}\left(\pi_i^{\alpha}\, v_i^{\alpha} + \sigma_{ki}^{\alpha}\,\frac{\partial v_i^{\alpha}}{\partial x_k}\right) \geqq 0 \quad , \tag{2.21}$$

where

$$\Theta = \sum_{\alpha=1}^{\nu}\frac{\partial}{\partial x_k}\left(\rho_{\alpha}\, u_k^{\alpha}\, A_{\alpha}\right) \quad , \quad \overset{*}{q}_k = q_k + T\sum_{\alpha=1}^{\nu}\rho_{\alpha}\, u_k^{\alpha}\, S_{\alpha} \quad . \tag{2.22}$$

Also

$$\rho\,\frac{DA}{Dt} + \Theta = \sum_{\alpha=1}^{\nu}\left(\rho_{\alpha}\,\frac{\overset{\alpha}{D}A_{\alpha}}{Dt} + m_{\alpha}\, A_{\alpha}\right) \quad . \tag{2.23}$$

An examination of (2.21) suggests that we put

$$\pi_k^{\alpha} = \frac{\partial \phi^{\alpha}}{\partial x_k} + \bar{\pi}_k^{\alpha} \quad , \quad \sigma_{ki}^{\alpha} = \delta_{ki}\,\phi^{\alpha} + \bar{\sigma}_{ki}^{\alpha} \quad , \tag{2.24}$$

where

$$\phi^{\alpha} = \sum_{\beta=1}^{\nu}\frac{\rho_{\alpha}\rho_{\beta}}{\rho}\left(A_{\alpha} - A_{\beta}\right) = \rho_{\alpha}\left(A_{\alpha} - A\right) \quad , \quad \sum_{\alpha=1}^{\nu}\phi^{\alpha} = 0 \quad . \tag{2.25}$$

In view of (2.24)-(2.25), (2.21) reduces to

$$-\rho\left(\frac{DA}{Dt} + S\,\frac{DT}{Dt}\right) - \frac{\overset{*}{q}_k}{T}\frac{\partial T}{\partial x_k} + \sum_{\alpha=1}^{\nu}\left(\bar{\pi}_i\, v_i^{\alpha} + \bar{\sigma}_{ki}^{\alpha}\,\frac{\partial v_i^{\alpha}}{\partial x_k}\right) \geqq 0 \quad . \tag{2.26}$$

Moreover, using (2.20), (2.22), (2.24) and (2.25), the energy equation (2.13) becomes

$$\rho r - \frac{\partial q_k^*}{\partial x_k} - \rho\left(\frac{DA}{Dt} + S\,\frac{DT}{Dt} + T\,\frac{DS}{Dt}\right) + \sum_{\alpha=1}^{\nu}\left(\bar{\pi}_i^\alpha\, v_i^\alpha + \bar{\sigma}_{ki}^\alpha\,\frac{\partial v_i^\alpha}{\partial x_k}\right) = 0 \quad . \tag{2.27}$$

We observe that the parts of the partial stresses and diffusive forces which depend on ϕ^α do not contribute to the equations of motion (2.15), the total stress $\sum_{\alpha=1}^{\nu} \sigma_{ki}^\alpha$ and the energy equation (2.27).

Since

$$\sum_{\alpha=1}^{\nu} \rho_\alpha\, u_k^\alpha\, S_\alpha = \sum_{\alpha=1}^{\nu} \rho_\alpha\, u_k^\alpha (S_\alpha - S_\nu) \quad ,$$

we see that S_α occurs in the basic equations of the theory only in the combinations

$$S \quad , \quad S_\alpha - S_\nu \qquad (\alpha = 1,2,\ldots,\nu-1) \quad , \tag{2.28}$$

and of these only S appears in the entropy inequality. Moreover, $S_\alpha - S_\nu$ occurs only in the expression (2.22) for q_k and does not contribute to the equations of motion and energy, or to the partial stresses and diffusive forces. They must be specified by constitutive equations.

3. Mixture of elastic continua

We restrict attention to the case when the mass of each constituent is conserved so that

$$m_\alpha = 0 \quad . \tag{3.1}$$

Two methods seem to be available for discussing constitutive equations. In the first we assume that A_α, σ^α_{ki} $(\alpha = 1,..,\nu)$, π^α_i $(\alpha = 1,..,\nu-1)$, (and hence π^ν_i), S and $\overset{*}{q}_k$ are functions of

$$T \ , \ F^\beta_{ij} \ , \ \frac{\partial F^\beta_{ij}}{\partial X^\beta_k} \quad (\beta = 1,..,\nu) \quad , \tag{3.2}$$

and linear functions of degree one in

$$T_{,k} \ , \ v^\beta_k - v^\nu_k \quad (\beta = 1,..,\nu-1) \quad . \tag{3.3}$$

We might regard a mixture of ν elastic continua as one whose constitutive equations involve nonlinear functions of the quantities in (3.3) as well as those in (3.2), but we restrict our attention here to functions linear in the variables (3.3). If we use these constitutive assumptions in the inequality (2.21) we can show that A and S reduce to functions of

$$T \ , \ F^\beta_{ij} \quad , \tag{3.4}$$

and that restrictions are placed on A_α and other quantities. In the

general case of ν constituents there is some algebraic complexity in making explicit deductions about the form of A_α from these restrictions, although some information can be obtained about the "equilibrium" values of A_α, i.e., when $v_k^\beta - v_k^\nu$ vanish.

In the second method we assume that A_α (and hence A and ϕ^α), S, $\bar{\sigma}_{ki}^\alpha$ $(\alpha = 1,..,\nu)$, q_k^*, $\bar{\pi}_i^\alpha$ $(\alpha = 1,..,\nu-1)$, and hence $\bar{\pi}_i^\nu$, are functions of the variables in (3.2) and linear functions of degree one in the variables (3.3). We use the inequality (2.26) to place restrictions on these assumptions and again we find that A and S reduce to functions of the quantities (3.4). Inspection of (2.24) shows that while σ_{ki}^α will then depend on the functions (3.2) and also depend linearly of degree one on (3.3), the quantities π_k^α are in addition dependent on

$$\frac{\partial^2 T}{\partial x_r\, \partial x_k} \, , \quad \frac{\partial^2 F_{ij}^\beta}{\partial X_k^\beta\, \partial X_r^\beta} \, , \quad v_{i,k}^\beta - v_{i,k}^\nu \, , \tag{3.5}$$

if we ignore, for the moment, any restrictions which might arise from (2.21). We add the additional restriction that π_k^α is a function only of the variables (3.2) and a linear function of degree one of the variables (3.3). With this extra condition, coupled with the restrictions already found on A, it can be shown that A_α (and hence ϕ^α) reduce to functions of the variables (3.4). Then σ_{ki}^α, π_k^α depend on the same variables (3.2) and (3.3) as in the first method. The remaining restriction can be found from (2.21) (with (2.23)), or from (2.26). We use the second method here which is slightly more restrictive than the first, and we quote the final results[+]. Thus, in addition to the result that A_α is a function of the variables (3.4), we have

[+]There is, however, no essential difficulty in adopting the first procedure.

$$S = -\sum_{\alpha=1}^{\nu} \frac{\rho_\alpha}{\rho} \frac{\partial A_\alpha}{\partial T} = -\frac{\partial A}{\partial T} \quad , \tag{3.6}$$

$$\sigma^{\alpha}_{ki} = \sum_{\beta=1}^{\nu} \rho_\beta \frac{\partial A_\beta}{\partial F^{\alpha}_{ij}} F^{\alpha}_{kj} \quad , \tag{3.7}$$

$$\pi^{\alpha}_{k} = \sum_{\beta=1}^{\nu} \left(\rho_\alpha \frac{\partial A_\alpha}{\partial F^{\beta}_{ij}} \frac{\partial F^{\beta}_{ij}}{\partial X^{\beta}_{m}} \frac{\partial X^{\beta}_{m}}{\partial x^{\beta}_{k}} - \rho_\beta \frac{\partial A_\beta}{\partial F^{\alpha}_{ij}} \frac{\partial F^{\alpha}_{ij}}{\partial X^{\alpha}_{m}} \frac{\partial X^{\alpha}_{m}}{\partial x^{\alpha}_{k}}\right)$$

$$+ \sum_{\beta=1}^{\nu} c^{\alpha\beta}_{km} (v^{\beta}_{m} - v^{\nu}_{m}) + b^{\alpha}_{km} \frac{\partial T}{\partial x_m} \quad , \tag{3.8}$$

$$q^{*}_{k} = D_{km} \frac{\partial T}{\partial x_m} + \sum_{\beta=1}^{\nu} D^{\beta}_{km} (v^{\beta}_{m} - v^{\nu}_{m}) \quad , \tag{3.9}$$

where

$$\sum_{\alpha=1}^{\nu} c^{\alpha\beta}_{km} = 0 \quad , \quad \sum_{\alpha=1}^{\nu} b^{\alpha}_{km} = 0 \quad , \tag{3.10}$$

and the coefficients in (3.8) and (3.9) are functions of the quantities in (3.2). These coefficients satisfy inequalities arising from a quadratic inequality in $T_{,k}$ and $v^{\beta}_{k} - v^{\nu}_{k}$ $(\beta = 1,\ldots,\nu-1)$ but these are not recorded here. Finally, we note that the above constitutive equations are subject to the usual invariance conditions under superposed rigid body motions of the whole mixture. For example A_α reduces to the new form

$$A_\alpha = A_\alpha\left(\frac{\partial x^{\alpha}_{m}}{\partial X^{\alpha}_{i}} \frac{\partial x^{\beta}_{m}}{\partial X^{\beta}_{j}}, T\right) \qquad (\beta = 1,\ldots,\nu) \quad . \tag{3.11}$$

We observe that if we had adopted the first procedure outlined at the beginning of this section the final constitutive equations would have been more complicated. It is still possible to add further terms to the constitutive equations of the type discussed by Green and Naghdi [6], but these yield partial stress contributions which are not necessarily derivable from the energy functions A_α and are omitted here. Such terms would influence the values of the partial stresses and diffusive forces, but would make no contribution to basic equations of motion and energy or to the total stress.

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13. ABSTRACT

For a mixture of chemically inert elastic continua, constitutive equations are discussed which are nonlinear functions of temperature and kinematical variables for each constituent but are linear functions of degree one in temperature gradient and velocity differences.

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